1. Basic principles of Elasticity and plasticity

2. Stress and Deformation of Bars in Axial load
Basic principles of Elasticity and plasticity

**Elasticity and plasticity in building engineering** – theoretical basement for the theory of structures (important for steel, concrete, timber structures design) - to be able design safe structures (to resist mechanical load, temperature load…)

**Statics**: external forces, internal forces

**Elasticity and plasticity new terms:**
1) stress
2) strain
3) stability
Principal terms

- Stress
- Strain
- Stability

Material

- Elastic behavior of material – *Hooke’s law - Elasticity*
- Plastic behavior of material - *Plasticity*

Load

- External force load \((F, M, q)\)
- Temperature load
Stress and Strain

Elasticity and plasticity - new terms stress and strain

External forces, internal forces, stress

\[ \Delta \vec{N} \quad \text{... Normal component of the vector} \]
\[ \Delta \vec{V} \quad \text{... Tangential component of } \Delta \vec{F} \]
\[ \Delta A \quad \text{... Element of cross section area } A \]

\[
\sigma = \lim_{\Delta A \to 0} \frac{\Delta \vec{N}}{\Delta A}
\]

\[ \tau = \lim_{\Delta A \to 0} \frac{\Delta \vec{V}}{\Delta A} \]

Normal component of \( \Delta \vec{F} \)

Stress (intensity of the internal forces distributed over a given section)
The initial presumptions of the classic linear elasticity:

1) continuity of material,
2) homogeneity (just one material) and isotropy (properties are the same in all directions),
3) linear elasticity (valid Hook’s law),
4) the small deformation theory,
5) static loading,
6) no initial state of stress
1. **Continuity of material:**
A solid is a continuum, it has got its volume without any holes, gaps or any interruptions. Stress and strain is a continuous function.

2. **Homogenity and isotropy**
3. **Linear elasticity**
4. **Small deformation**
5. **Static loading**
6. **No initial state of stress**
1. Continuity of material

2. Homogenity a isotropy:

   **Homogeneous** material has got physical characteristics identical in all places (concret, steel, timber).
   Combination of two or more materials (concret + steel) is not homogeneous material.

   **Isotropy** means that material has got characteristics undependent on the direction – (concret, steel – yes, timber – not).

3. Linear elasticity
4. Small deformation
5. Static loading
6. No initial state of stress
Basic principles of Elasticity and plasticity

1. Continuity of material
2. Homogenity and isotropy

3. Linear elasticity:
   Elasticity is an ability of material to get back after removing the causes of changes (for example load) into the original state. If there is a direct relation between stress and strain than we talk about Hooke’s law = this is called physical linearity.

4. Small deformation
5. Static loading
6. No initial state of stress
Plasticity (on the contrary from linearity): This is an ability of material to deform without any rupture by non-returnable way. After removing the load there are staying permanent deformations.
**Basic principles of Elasticity and plasticity**

For axial load

\[ \varepsilon \approx \frac{\Delta l}{l} \]

Stress-strain diagram of an ideal elastic-plastic material:

\[ \sigma = \varepsilon E \]

\[ \varepsilon_{\text{elast.}} \]

\[ \varepsilon_{\text{plast.}} \]

**Compression**

\[ \varepsilon_x = \frac{\Delta dx}{dx} \]

\[ \varepsilon \ll 1 \]

Small-deformation theory

**Tension**

\[ +\varepsilon = \Delta l/l \]

\[ \sigma - \text{normal stress} \]

\[ \varepsilon - \text{strain} \]
**Hooke’s law** - physical relations between stress and strain.

It is valid only in the linear elastic range of the stress-strain diagram.

\[ \sigma_x = \varepsilon_x \cdot E \]

**Basic principles of Elasticity and plasticity**

**Hooke’s law**, another expression

- By substituting:
  \[ \sigma_x = \frac{N}{A} \]
  \[ \varepsilon_x = \frac{\Delta l}{l} \]

- Axial strain [-]

- Normal stress [Pa]

- Young’s modulus of elasticity in tension and compression [Pa]
Hooke’s law in shear

\[ \tau_{xz} = \gamma_{xz} G \]

\( \gamma_{xz} \) ... Angle deformation

\( \tau_{xz} \) ... Shear stress [Pa]

\( G \) ... modulus of elasticity in shear [Pa]

\[ \tan \alpha = \frac{\tau}{\gamma} = G \]

\[ \alpha = \arctg G \]

Small-deformation theory:

\( \gamma << 1 \)

Simplyfying:

\[ \tan \gamma \approx \gamma \]
Physical constants – $E$, $G$, $\nu$

In case of isotropic materials $E$, $G$ are not mutual independent.

$$\frac{E}{G} = 2(1+\nu)$$

Poisson’s ratio

$$0 \leq \nu \leq 0.5 \quad \rightarrow \quad \frac{E}{3} \leq G \leq \frac{E}{2}$$

Benchmark values of physical constants of some materials:

<table>
<thead>
<tr>
<th>Material</th>
<th>$E$</th>
<th>$G$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>210 000 MPa</td>
<td>81 000 MPa</td>
<td>0.3</td>
</tr>
<tr>
<td>Glass</td>
<td>70 000 MPa</td>
<td>28 000 MPa</td>
<td>0.25</td>
</tr>
<tr>
<td>Granite</td>
<td>12 000 to 50 000 MPa</td>
<td>-</td>
<td>0.2</td>
</tr>
<tr>
<td>Softwood</td>
<td>$E_{\parallel} = 10 000$ MPa, $E_{\perp} = 300$ MPa</td>
<td>600 MPa</td>
<td>-</td>
</tr>
</tbody>
</table>
**Basic principles of Elasticity and plasticity**

**Stress-strain diagrams**

(a) Stress-strain diagram for concrete

- $f_u$: Ultimate limit
- $0.4f_u$: Elastic limit
- $arctgE$: Young's modulus
- $\varepsilon_{cu}$: Ultimate strain
- $\varepsilon_{c1}$: Elastic strain

(b) Stress-strain diagram for steel

- $f_e$: Elasticity limit
- $f_y$: Yield limit
- $f_u$: Ultimate limit
- $\sigma$: Stress
- $\varepsilon$: Strain

**Plasticity**: the ability of material to get permanent deformations without fracture

**Ductility**: plastic elongation of a broken bar (range /OT/), steel 15%.
1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity

4. Small deformations theory:
Changes of a shape of a (solid) structure are small with aspect to its size (dimensions). Than we can use a lot of mathematical simplifications, which usually lead to linear dependency.

5. Static loading
6. No initial state of stress
Basic principles of Elasticity and plasticity

Theory of small deformations

Theory of the I-st order

\[ F \]
\[ \delta \ll l \]
\[ H \]
\[ b \]
\[ l \]
\[ M_{ay} = H \cdot l \]

Theory of the II-nd order - Geometric nonlinearity

Theory of large deformation (finite deformation)

\[ F \]
\[ \delta \approx l \]
\[ H \]
\[ b \]
\[ \delta \]
\[ l \]
\[ M_{ay} = H \cdot l + F \cdot \delta \]

The equilibrium conditions we set on the deformed construction (buckling of columns).
1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity
4. Small deformations theory

5. Static loading:
   It means gradually growing of load (not dynamic effects)

6. No initial state of stress
Basic principles of Elasticity and plasticity

1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity
4. Small deformations theory
5. Static loading

6. No initial state of stress:
   In the initial state there are all stresses equal zero. (Inner tension e.g. from the production).
Basic principles of Elasticity and plasticity

1. Continuity of material
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All these assumptions enable to use principal of superposition which is based on linearity of all mathematic relationship.
Saint - Venant principle of local effect

Jean Claude Saint-Venant (1797-1886)

Makes possible to replace a given load by a simpler one for the purpose of an easier calculation of stresses in a member.

- the state of stress is influenced just in near surroundings of the load
- farther from this load we have nearly uniform distribution of stress

Used for:
- replacement the surface load by the load statically equivalent but simpler for solution
Saint - Venant principle of local is not valid in these cases:

a) concentrated loads on the end of bar:

\[ \sigma_x = \text{const.} \quad \sigma_x \neq \text{const.} \]

b) bars with variable cross-section area: deduced conditions are valid for bars with gradual changes of cross-section area. Abrupt changes (announce by holes, nicks or narrowing) lead to no validity of condition.
These theorems are valid when the basic principles are kept.

Basic Theorems of Statics

1) Principle of action and reaction
2) Principle of superposition
3) Principle of proportionality
Stress and Strain

Elasticity and plasticity - new terms **stress and strain**

External forces, internal forces, stress

\[ \Delta \vec{N} \quad \text{... Normal component of the vector} \]

\[ \Delta \vec{V} \quad \text{... Tangential component of } \Delta \vec{F} \]

\[ \Delta A \quad \text{... Element of cross section area } A \]

\[ \sigma = \lim_{\Delta A \to 0} \frac{\Delta \vec{N}}{\Delta A} \]

**stress** *(intensity of the internal forces distributed over a given section)*

\[ \tau = \lim_{\Delta A \to 0} \frac{\Delta \vec{V}}{\Delta A} \]

**shear**
Stress: vector, characterised by its components

Unit: Pascal ... [Pa]

\[ \text{Pa} = \frac{\text{N}}{\text{m}^2} \]

The Pascal is a small quantity, in practice we use multiplies of this unit

\[ \text{MPa} = 10^6 \text{Pa} = \frac{\text{MN}}{\text{m}^2} = \frac{\text{N}}{\text{mm}^2} \]

\[ \text{kPa} = 10^3 \text{Pa} = \frac{\text{kN}}{\text{m}^2} \]
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Normal force $N \neq 0$
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Bending moment $M_y, M_z \neq 0$
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Torsion moment: \( M_x \neq 0 \)

Basic (simple) types of mechanical stress

\[ n_v = 6 \]
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Shear force $V_y$, $V_z \neq 0$

Diagram showing forces and shear forces at points $a$ and $b$.
### Basic (simple) types of mechanical stress

<table>
<thead>
<tr>
<th>Type of the Loading</th>
<th>Internal Force</th>
<th>Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial Loading</td>
<td>$N$</td>
<td>$\sigma_x$ - normal (tension, compression)</td>
</tr>
<tr>
<td>Bending</td>
<td>$M_y, M_z$</td>
<td>$\sigma_x$ - normal (bending stress)</td>
</tr>
<tr>
<td>Shear</td>
<td>$V_y, V_z$</td>
<td>$\tau_{xy}, \tau_{xz}$ - shear</td>
</tr>
<tr>
<td>Torsion</td>
<td>$M_x$</td>
<td>$\tau_{xy}, \tau_{xz}$ - shear</td>
</tr>
</tbody>
</table>
Types of loading

a) simple (axial loading, bending, torsion, shear)
b) combined

Combined loading:

- general bending (unsymmetric bending)
- eccentric axial loading
- torsion combined with tension or compression and with bending

Due to the Principle of superposition, which is valid in a linear elastic range, we can solve the combined stresses. First by splitting up to basic stresses and then we can add these results together.
Axial loading – tension, compression

The only one inner force in each cross-section is an axial force $N$.

$$M_x = 0 \quad V_y = V_z = 0 \quad M_y = M_z = 0$$

$N > 0$ … tension

$N < 0$ … compression
Conditions of solution

a) deformated cross-sections stay on plane figure and it is vertically to the axis (Bernoulli hypothesis)

Character of condition is deformation-geometrical. Cross sections stay mutually parallel without tapering.

\[ \gamma_{xy} = \gamma_{xz} = 0 \rightarrow \tau_{xy} = \tau_{xz} = 0 \]

\[ \sigma_x = \text{const.} \]

b) longitudinal fibres are not mutually compressed together

\[ \sigma_y = \sigma_z = 0 \]
Stress and strain in external load

Axis x = axis of a member

- Intensity of internal forces distributed over a given section
- Tensile stress - positive sign
  compressive stress – negative sign

a) Stress under axial loading

axial force F → normal forces N → normal stress $\sigma_x$

$$\sigma_x = \frac{N}{A}$$

[MPa]
b) Strain under axial loading - deformation

*Deformations: elongation or contraction*

Axial strain

\[ \varepsilon_x = \frac{\Delta l}{l} \]

Lateral strain

\[ \varepsilon_y = \varepsilon_z = -\nu \varepsilon_x \]

\[ \nu \leq 0.5 \quad \text{Poisson's ratio} \]

*Dimension's changes:*

\[ l' = l + \Delta l \]

\[ h' = h + \Delta h \]

\[ b' = b + \Delta b \]

Circle - diameter \( d \)?
Stress and strain in temperature changes

a) stress

If there is not defended the deformation of a member – doesn’t come up normal force and stress, later on indeterminate members

b) Strain (thermal strain)

\[ \varepsilon_{xT} = \varepsilon_{yT} = \varepsilon_{zT} = \alpha_T \Delta T \]

\[ \alpha_T \] - Coefficient of thermal expansion \([°C^{-1}]\)

\[ \Delta l = \alpha_T \cdot \Delta T \cdot l \]

\[ \varepsilon_{xT} = \Delta l/l = \Delta b/b = \Delta h/h = \Delta d/d \]
Example 1

The steel rod (see the picture) has a circle cross-sectional area of a diameter d = 0.025 m. E = 2,1.10^5 MPa. ν = 0.3

Determine σ_x, elongation of the rod, the lateral changes (in dimensions) and determine new dimensions of the rod). (Ignore the dead weight).

Results:
A = 490,87.10^{-6} m^2
σ_x = 203,718 MPa

Δl = 0.0097 m = 9.7.10^{-3} m = 9.7 mm
ε_x = 9.7.10^{-4}
l’ = 10,0097 m

ε_y = ε_z = -2,91.10^{-4}
Δd = -7.28.10^{-6} m = -7.28.10^{-3} mm
d’ = 0.02499 m

P = 100 kN

R

N

l = 10 m
Determine the total deformation of the rod in its length (see the picture).
\( d_1 = 20 \text{mm}, \ d_2 = 10 \text{mm}, \ F_1 = 40 \text{kN}, \ F_2 = 10 \text{kN}, \ l_1 = 3 \text{m}, \ l_2 = 2 \text{m}, \ E = 2,1.10^5 \text{Mpa}. \)

\[ R = F_1 - F_2 = 40 - 10 = 30 \text{ kN} \]

\[ N_1 = -R = -30 \text{kN} \]

\[ N_2 = -R + F_1 = -30 + 40 = 10 \text{ kN} \]

Result: \( \Delta l = -0,152 \text{mm}. \)

Results:
\[ A_1 = 314,159.10^{-6} \text{m}^2 \]
\[ A_2 = 78,539.10^{-6} \text{m}^2 \]
\[ \Delta l_1 = -1,364.10^{-3} \text{m} = -1,364 \text{mm} \]
\[ \Delta l_2 = 1,212.10^{-3} \text{m} = 1,212 \text{mm} \]
\[ \sigma_1 = -95,49 \text{MPa} \]
\[ \sigma_2 = 127,33 \text{MPa} \]
The concrete column of a square cross-section of a side $a = 0.6\text{ m}$ and the height $h = 3.6\text{ m}$ is uniformly warmed by $\Delta T = 75^\circ\text{C}$. Determine the changes in dimensions of the column - cube. $\alpha_T = 10 \cdot 10^{-6} \text{ °C}^{-1}$

Results: $h' = 3.6027\text{ m}$, $a' = 0.6005\text{ m}$
Determine the stress in the circle rod and determine the change of the length of the rod. 

\( F = 20 \text{kN}, \ \Delta t = 15^\circ \text{C}, \ d = 0.02 \text{m}, \ l = 1.5 \text{m}, \ l_1 = 1 \text{ m}, \ l_2 = 2 \text{ m}, \ E = 210 \text{GPa}, \ \alpha_T = 0.000012^\circ \text{C}^{-1} \)

**Solution:**

1. N force from \( F \)
2. normal stress from \( N \)
3. elongation of the rod (from \( N \) + from temperature)

**Results:**

\( N = 30 \text{kN}, \ \sigma_x = 95.5 \text{MPa (tension)}, \ \Delta l_N = 0.682 \text{mm}, \ \Delta l_T = 0.27 \text{mm} \)