1. Basic principles of Elasticity and plasticity

2. Stress and Deformation of Bars in Axial load
Basic principles of Elasticity and plasticity

Elasticity and plasticity in building engineering – theoretical basement for the theory of structures (important for steel, concrete, timber structures design) - to be able design safe structures (to resist mechanical load, temperature load…)

Statics: external forces, internal forces

Elasticity and plasticity new terms:
1) stress
2) strain
3) stability
Principal terms

- Stress
- Strain
- Stability

**Material**
- Elastic behavior of material – *Hooke’s law - Elasticity*
- Plastic behavior of material - *Plasticity*

**Load**
- External force load \((F, M, q)\)
- Temperature load
Basic principles of Elasticity and plasticity

The initial presumptions of the classic linear elasticity:

1) continuity of material,
2) homogenity (just one material) and isotropy (properties are the same in all directions),
3) linear elasticity (valid Hook’s law),
4) the small deformation theory,
5) static loading,
6) no initial state of stress
1. **Continuity of material:**
A solid is a continuum, it has got its volume without any holes, gaps or any interruptions. Stress and strain is a continuous function.

2. **Homogenity and isotropy**
3. **Linear elasticity**
4. **Small deformation**
5. **Static loading**
6. **No initial state of stress**
Basic principles of Elasticity and plasticity

1. Continuity of material

2. Homogeneity and isotropy:
   Homogeneous material has got physical characteristics identical in all places (concrete, steel, timber). Combination of two or more materials (concrete + steel) is not homogeneous material.
   Isotropy means that material has got characteristics independent on the direction – (concrete, steel – yes, timber – not).

3. Linear elasticity
4. Small deformation
5. Static loading
6. No initial state of stress
1. Continuity of material
2. Homogeneity and isotropy

3. Linear elasticity:
   Elasticity is an ability of material to get back after removing the causes of changes (for example load) into the original state. If there is a direct relation between stress and strain than we talk about Hooke’s law = this is called physical linearity.

4. Small deformation
5. Static loading
6. No initial state of stress
Plasticity (on the contrary from linearity): This is an ability of material to deform without any rupture by non-returnable way. After removing the load there are staying permanent deformations.
Basic principles of Elasticity and plasticity

For axial load

Stress-strain diagram of an ideal elastic-plastic material:

- Yield limit $f_y$
- Tension
- Compression

$\sigma - \text{normal stress}$
$\varepsilon - \text{strain}$

$\sigma_x = \varepsilon_x \cdot E$

For linear elastic range:

$\text{arctg } E = \alpha$

$\varepsilon_{\text{elast.}}$ $\varepsilon_{\text{plast}}$

$+\varepsilon = \Delta l/l$
**Hooke’s law** - physical relations between stress and strain

\[ \sigma_x = \varepsilon_x \cdot E \]

\( E \) is Young’s modulus of elasticity in tension and compression [Pa].

\( \varepsilon \) is Axial strain [-].

\( f_y \) is Yield limit.

**Linear elastic range**

By substituting:

\[ \Delta l = \frac{Nl}{EA} \]

\( \tan \alpha = \frac{\sigma}{\varepsilon} = E \)
Hooke’s law in shear

\[ \tau_{xz} = \gamma_{xz} G \]

\[ \frac{E}{G} = 2(1 + v) \]

\( \gamma_{xz} \) … Angle deformation

\( \tau_{xz} \) … Shear stress [Pa]

\( G \) … modulus of elasticity in shear [Pa]
**Basic principles of Elasticity and plasticity**

*Stress-strain diagrams*

(a) **concrete**

- $\sigma(<0)$
- $f_u$
- $0.4f_u$
- $\arctg E$
- $\varepsilon_{c1}$
- $\varepsilon_{cu}$

(b) **steel**

- $\sigma$
- $f_u$
- $f_y$
- $f_{pr}$

**Obr. 1.11 Pracovní diagram (a) betonu v tlaku a (b) oceli v tahu**

**Plasticity:** the ability of material to get permanent deformations without fracture

**Ductility:** plastic elongation of a broken bar (range /OT/), steel 15%.
1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity

4. Small deformations theory:
   Changes of a shape of a (solid) structure are small with aspect to its size (dimensions). Than we can use a lot of mathematical simplifications, which usually lead to linear dependency.

5. Static loading
6. No initial state of stress
Basic principles of Elasticity and plasticity

Theory of small deformations

\[ \delta \ll l \]

Theory of the I-st order

\[ M_{ay} = H.l \]

Theory of large deformation (finite deformation)

\[ \delta \approx l \]

Theory of the II-nd order - Geometric nonlinearity

\[ M_{ay} = H.l + F.\delta \]

The equilibrium conditions we set on the deformed construction (buckling of columns).
Basic principles of Elasticity and plasticity

1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity
4. Small deformations theory

5. Static loading:
   It means gradually growing of load (not dynamic effects)

6. No initial state of stress
Basic principles of Elasticity and plasticity

1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity
4. Small deformations theory
5. Static loading

6. No initial state of stress:
   In the initial state there are all stresses equal zero.
   (Inner tension e.g. from the production).
1. Continuity of material
2. Homogenity and isotropy
3. Linear elasticity
4. Small deformations theory
5. Static loading
6. No initial state of stress

All these assumptions enable to use principal of superposition which is based on linearity of all mathematic relationship.
Saint - Venant principle of local effect

Jean Claude Saint-Venant (1797-1886)

Makes possible to replace a given load by a simpler one for the purpose of an easier calculation of stresses in a member.

- the state of stress is influenced just in near surroundings of the load
- farther from this load we have nearly uniform distribution of stress

Used for:
replacement the surface load by the load statically equivalent but simpler for solution
Saint - Venant principle of local is not valid in these cases:

a) concentrated loads on the end of bar:

\[ F \]

\[ F \]

\[ \sigma_x = \text{const.} \]

\[ \sigma_x \neq \text{const.} \]

b) bars with variable cross-section area: deduced conditions are valid for bars with gradual changes of cross-section area. Abrupt changes (announce by holes, nicks or narrowing) lead to no validity of condition.
Theorems of superposition and proportionality

These theorems are valid when the basic principles are kept.

Basic Theorems of Statics

1) Principle of action and reaction
2) Principle of superposition
3) Principle of proportionality
Stress and Strain

Elasticity and plasticity - new terms **stress and strain**

**External forces, internal forces, stress**

\[ \Delta \vec{N} \] ... Normal component of the vector

\[ \Delta \vec{V} \] ... Tangential component of \[ \Delta \vec{F} \]

\[ \Delta A \] ... Element of cross section area \( A \)

\[
\sigma = \lim_{\Delta A \to 0} \frac{\Delta \vec{N}}{\Delta \vec{A}}
\]

**stress** (*intensity of the internal forces distributed over a given section*)

\[
\tau = \lim_{\Delta A \to 0} \frac{\Delta \vec{V}}{\Delta \vec{A}}
\]

**shear**
Stress

Stress: vector, characterised by its components

Unit: Pascal ... [Pa]

The Pascal is a small quantity, in practise we use multiplies of this unit.

\[
\text{Pa} = \frac{N}{m^2}
\]

\[
\text{MPa} = 10^6 \text{Pa} = \frac{MN}{m^2} = \frac{N}{mm^2}
\]

\[
\text{kPa} = 10^3 \text{Pa} = \frac{kN}{m^2}
\]
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Normal force $N \neq 0$
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Bending moment $M_y, M_z \neq 0$

- compression
- tension
- compression
- tension
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Torsion moment $M_x \neq 0$

$n_v = 6$
Basic (simple) types of mechanical stress

1. Axial loading
2. Bending
3. Torsion
4. Shear

Shear force $V_y, V_z \neq 0$
### Basic (simple) types of mechanical stress

<table>
<thead>
<tr>
<th>Type of the loading</th>
<th>Internal force</th>
<th>Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial Loading</td>
<td>$N$</td>
<td>$\sigma_x$ - normal (tension, compression)</td>
</tr>
<tr>
<td>Bending</td>
<td>$M_y, M_z$</td>
<td>$\sigma_x$ - normal (bending)</td>
</tr>
<tr>
<td>Shear</td>
<td>$V_y, V_z$</td>
<td>$\tau_{xy}, \tau_{xz}$ - shear</td>
</tr>
<tr>
<td>Torsion</td>
<td>$M_x$</td>
<td>$\tau_{xy}, \tau_{xz}$ - shear</td>
</tr>
</tbody>
</table>
Types of loading

a) simple (axial loading, bending, torsion, shear)
b) combined

Combined loading:

- general bending (unsymmetric bending)
- eccentric axial loading
- torsion combined with tension or compression and with bending

Due to the Principle of superposition, which is valid in a linear elastic range, we can solve the combined stresses. First by splitting up to basic stresses and then we can add these results together.
Axial loading – tension, compression

The only one inner force in each cross-section is an axial force $N$.

\[
\begin{align*}
M_x &= 0 \\
V_y &= V_z = 0 \\
M_y &= M_z = 0
\end{align*}
\]

$N > 0$ … tension

$N < 0$ … compression
Conditions of solution

a) deformed cross-sections stay on plane figure and it is vertically to the axis (Bernoulli hypothesis)

Character of condition is deformation-geometrical. Cross sections stay mutually parallel without tapering.

\[ \tau_{xy} = \tau_{xz} = 0 \rightarrow \tau_{xy} = \tau_{xz} = 0 \]

\[ \sigma_x = \text{const.} \]

b) longitudinal fibres are not mutually compressed together

\[ \sigma_y = \sigma_z = 0 \]
Stress and strain in external load

Axis \( x = \text{axis of a member} \)

\[ \sigma_x = \frac{N}{A} \text{ [MPa]} \]

a) Stress under axial loading

axial force \( F \rightarrow \) normal forces \( N \rightarrow \) normal stress \( \sigma_x \)

- Intensity of internal forces distributed over a given section
- Tensile stress - positive sign
  compressive stress – negative sign
b) Strain under axial loading - deformation

**Deformations**: elongation or contraction

**Axial strain**

\[
\varepsilon_x = \frac{\Delta l}{l}
\]

*(dimensionless quantity [-]*)

**Lateral strain**

\[
\varepsilon_y = \varepsilon_z = -\nu \varepsilon_x
\]

\[
\nu \leq 0.5 \quad \text{Poisson's ratio}
\]

**Dimension's changes:**

\[
l' = l + \Delta l
\]

\[
h' = h + \Delta h
\]
Stress and strain in temperature changes

a) stress  If there is not defended the deformation of a member – doesn’t come up normal force and stress, later on indeterminate members

b) Strain (thermal strain)

\[ \varepsilon_{xT} = \varepsilon_{yT} = \varepsilon_{zT} = \alpha_T \Delta T \]

\[ \alpha_T \text{ - Coefficient of thermal expansion } [^\circ\text{C}^{-1}] \]

\[ \Delta l = \alpha_T \cdot \Delta T \cdot l \]

\[ \varepsilon_{xT} = \Delta l/l = \Delta b/b = \Delta h/h = \Delta d/d \]
Example 1

The steel rod (see the picture) has a circle cross-sectional area of a diameter $d = 0.025 \text{ m}$. $E = 2.1 \times 10^5 \text{MPa}$. $\nu = 0.3$

Determine $\sigma_x$, elongation of the rod, the lateral changes (in dimensions) and determine new dimensions of the rod). (Ignore the dead weight).

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**Results:**

- $A = 490.87 \times 10^{-6} \text{m}^2$
- $\sigma_x = 203.718 \text{MPa}$

- $\Delta l = 0.0097 \text{m} = 9.7 \times 10^{-3} \text{m} = 9.7 \text{mm}$
- $\varepsilon_x = 9.7 \times 10^{-4}$
- $l' = 10.0097 \text{m}$

- $\varepsilon_y = \varepsilon_z = -2.91 \times 10^{-4}$
- $\Delta d = -7.28 \times 10^{-6} \text{m} = -7.28 \times 10^{-3} \text{mm}$
- $d' = 0.02499 \text{m}$

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$P = 100 \text{kN}$
Example 2

Determine the total deformation of the rod in its length (see the picture).

\( d_1 = 20\text{mm}, \ d_2 = 10\text{mm}, \ F_1 = 40\text{kN}, \ F_2 = 10\text{kN}, \ l_1 = 3\text{m}, \ l_2 = 2\text{m}, \ E = 2,1 \times 10^9\text{Mpa} \).

\[ \Sigma \text{F}_{\text{ix}} = 0: \ R - F_1 + F_2 = 0 \]
\[ R = F_1 - F_2 = 40 - 10 = 30\text{ kN} \]
\[ N_1 = -R = -30\text{kN} \]
\[ N_2 = -R + F_1 = -30 + 40 = 10\text{ kN} \]

Result: \( \Delta l = -0,152\text{mm} \).

\[ A_1 = 314,159.10^{-6}\text{m}^2 \]
\[ A_2 = 78,539.10^{-6}\text{m}^2 \]
\[ \Delta l_1 = -1,364.10^{-3}\text{m} = -1,364\text{mm} \]
\[ \Delta l_2 = 1,212.10^{-3}\text{m} = 1,212\text{mm} \]
\[ \sigma_1 = -95,49\text{MPa} \]
\[ \sigma_2 = 127,33\text{MPa} \]
Example 3

The concrete column of a square cross-section 0.6 x 0.6 m and a height \( h = 3.6 \) m is uniformly warmed by \( \Delta T = 75^\circ C \). Determine the changes in dimensions of the column-cube.

\[ \alpha_T = 10 \cdot 10^{-6} \, ^\circ C^{-1} \]

Results: \( h' = 3.6027 \) m, \( a' = 0.6005 \) m, \( b' = 0.6005 \) m
Example 4

Determine the stress in the circle rod, draw its behaviour during the cross section and determine the change of the length of the rod.

\( F = 20 \text{kN} \quad \Delta t = 15^\circ \text{C} \)
\[ d = 0.02 \text{m} \quad l = 1.5 \text{m} \]
\[ l_1 = 1 \text{ m} \quad l_2 = 2 \text{ m} \]
\[ E = 210 \text{GPa} \quad \alpha_T = 0.000012^\circ \text{C}^{-1} \]

Solution:
1- N force from \( F \)
2- normal stress from \( N \)
3- elongation of the rod (from \( N \) + from temperature)

Results: \( N = 30 \text{kN}, \sigma_x = 95.5 \text{MPa (tension)}, \Delta l_N = 0.682 \text{mm}, \Delta l_T = 0.27 \text{mm} \)