Elastic-plastic stress analysis of prismatic bar under bending

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Ideal elastic-plastic material

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<td>Hook’s law</td>
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<tr>
<td>Y-A</td>
<td>Plastic state – free increase of deformations</td>
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<td>A-B</td>
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Stress-strain diagram

Compression

Tension

\[ \alpha = \arctan E \]

\[ \varepsilon_p \ldots \text{plastic (permanent) deformation} \]

\[ \varepsilon_e \ldots \text{elastic deformation} \]
Elastic stress analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

$\sigma_x$ stress behavior

$\sigma_{x, max} = \frac{M_{Ed}}{W_y}$

Stress-strain diagram

$\sigma_x$ vs. $\varepsilon_x$

compression

tension

$M_{Ed}$

state I.

Bending (direct) stress in outer fibres

$|\sigma_{x, max}| < f_{yd}$

$M_{Ed} \leq M_{Rd,el} = W_y \cdot f_{yd} = \frac{1}{6} b h^2 f_{yd}$

$M_{yd} = \frac{f_{yk}}{\gamma_M}$

Elasto-plastic stress analysis of prismatic bar under bending
Elastic stress analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

\[ \sigma_x \] stress behavior

\[ M_{Rd,el} = W_y \cdot f_{yd} = \frac{1}{6} b h^2 \cdot f_{yd} \]

Stress-strain diagram

\[ \sigma_x \]

state II.

Bending (direct) stress in outer fibres

\[ \sigma_{x,\text{max}} = f_{yd} \]

\[ f_{yd} = \frac{f_{yk}}{\gamma_M} \]

Elasto-plastic stress analysis of prismatic bar under bending
Elasto-plastic stress analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Load increases permanently, plastic zones come into being: 1 – under tension, 2 – under compression. Cross-section residuum has still elastic behavior.

Bernoulli hypothesis is still valid, \( \varepsilon_x \) is continuously linear. In point is

\[
\varepsilon_x = \frac{f_{yd}}{E}
\]
Elasto-plastic stress analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

\[ \sigma_x \text{ stress behavior} \]

\[ -f_{yd} \]

compression

tension

\[ M_{Rd,pl} \]

neutral axis is in the middle of cross-section area (in case of nonsymmetric cross-sections is moved under plastization)

Cross-section is plastized, so-called **plastic joint** comes into being, carrying capacity under bending is exhausted.

**state IV.**

\[ N = \int_A \sigma_x dA = 0 \quad \rightarrow \quad \int_{A_1} f_{yd} dA + \int_{A_2} -f_{yd} dA = f_{yd} \cdot (A_1 - A_2) = 0 \]

\[ A_1, A_2 \ldots \text{ cross-section areas in plastic state.} \]

**Neutral axis is in the middle of cross-section area.**

(in case of nonsymmetric cross-sections is moved under plastization)

\[ A_1 = A_2 = \frac{A}{2} \]
Elasto-plastic stress analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

\[ \sigma \] stress behavior

Compression

Tension

plastized

Cross-section is plastized, so-called **plastic joint** comes into being, carrying capacity under bending is exhausted.

**state IV.**

\[
M_{Rd,pl} = \int_{A} \sigma_x \cdot z \cdot dA = \int_{A_1} f_{yd} \cdot z \cdot dA + \int_{A_2} -f_{yd} \cdot z \cdot dA = f_{yd} \cdot (S_{1y} - S_{2y})
\]

\[
S_{1y} = -S_{2y} \quad \Rightarrow \quad S_{1y} + S_{2y} = 0 \quad \Rightarrow \quad M_{Rd,pl} = f_{yd} \cdot 2 \cdot S_{1y} = f_{yd} \cdot W_{y,pl}
\]

\[
W_{y,pl} = 2 \cdot S_{1y} \quad \ldots \text{plastic modulus of cross-section} \quad [\text{m}^3]
\]
Elasto-plastic stress analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

\[
\sigma_x \text{ stress behavior}
\]

\[
\begin{align*}
\sigma_x &= f_{yd} & \text{compression} \\
\sigma_x &= -f_{yd} & \text{tension}
\end{align*}
\]

Specifically:

\[
S_{1y} = A_1 \cdot \frac{h}{4} = \frac{h}{2} \cdot \frac{b}{4} = \frac{1}{8} b h^2
\]

\[
W_{y,pl} = 2 S_{1y} = 2 \cdot \frac{1}{8} b h^2 = \frac{1}{4} b h^2
\]

Plastic reserve of rectangular cross-section

\[
\frac{W_{y,pl}}{W_{y,el}} = \frac{\frac{1}{4} b h^2}{\frac{1}{6} b h^2} = \frac{6}{4} = 1.5 \rightarrow 50\%
\]

Elasto-plastic stress analysis of prismatic bar under bending
Example 1

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Assign of example: Determine $W_{y,\text{el,pl}}$ for cross-section with plastized outer quarters

Solution:

$$W_{y,\text{el,pl}} = W_{y,\text{el}} + W_{y,\text{pl}} = \frac{1}{6} b \left( \frac{h}{2} \right)^2 + 2 \left[ b \cdot \frac{h}{4} \left( \frac{h}{4} + \frac{h}{8} \right) \right] = \frac{1}{24} b \cdot h^2 + \frac{3}{16} b \cdot h^2$$

Result:

$W_{y,\text{el,pl}} = \frac{11}{48} b \cdot h^2$

(valid for plastized outer quarters of cross-section only!!!)

<table>
<thead>
<tr>
<th>state</th>
<th>$W_y$ [m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.</td>
<td>$\frac{1}{6} b \cdot h^2$</td>
</tr>
<tr>
<td>III.</td>
<td>$\frac{11}{48} b \cdot h^2$</td>
</tr>
<tr>
<td>IV.</td>
<td>$\frac{1}{4} b \cdot h^2$</td>
</tr>
</tbody>
</table>

Elasto-plastic stress analysis of prismatic bar under bending
Example 2.1

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Assign of Determine maximal load-carrying capacity of beam $q_d [\text{kN/m}]$ for example: condition of solution: a) maximal bending (direct) stress $\sigma_x = f_{yd}$

Input values: $l = 6 \text{m}$ $b = 20 \text{mm}$ $h = 80 \text{mm}$ $f_{yk} = 235 \text{MPa}$ $\gamma_{M0} = 1,15$

Solution:

$$M_{Ed,el} = M_{y,\text{max}} = \frac{1}{8} q_{d,el} l^2$$

$$W_{y,el} = \frac{1}{6} b h^2 = 2,13 \times 10^{-5} \text{m}^3$$

$$f_{yd} = \frac{f_{yk}}{\gamma_{M0}} = 204,35 \text{MPa}$$

$$\sigma_{x,\text{max}} = f_{yd} = \frac{M_{sd,el}}{W_{y,el}} \quad \rightarrow \quad f_{yd} = \frac{q_{d,el} l^2}{8 W_{y,el}} \quad \rightarrow \quad q_{d,el} = \frac{8 f_{yd} W_{y,el}}{l^2} = 0,97 \text{kN/m}$$

Elasto-plastic stress analysis of prismatic bar under bending
Example 2.2

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

**Assign of Example:** Determine maximal load-carrying capacity of beam \( q_d \) [kN/m] for condition of solution: b) outer quarters of cross-section are plastized

**Input values:** \( l = 6\,\text{m} \), \( b = 20\,\text{mm} \), \( h = 80\,\text{mm} \), \( f_{yk} = 235\,\text{MPa} \), \( \gamma_{M0} = 1.15 \)

**Solution:**

\[
M_{Ed, el, pl} = \frac{1}{8} q_d \cdot l^2
\]

\[
W_{y, el, pl} = \frac{11}{48} b \cdot h^2 = 2.93 \times 10^{-5} \, \text{m}^3
\]

\[
f_{yd} = \frac{f_{yk}}{\gamma_{M0}} = 204.35\,\text{MPa}
\]

\[
\sigma_{x, max} = f_{yd} = \frac{M_{sd, el, pl}}{W_{y, el, pl}} = \frac{q_d \cdot l^2}{8 \cdot W_{y, el, pl}} \rightarrow \frac{q_d \cdot l^2}{W_{y, el, pl}} = 1.33\,\text{kN/m}
\]
Example 2.3

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Assign of example: Determine maximal load-carrying capacity of beam $q_d$ [kN/m] for condition of solution: c) plastic joint comes into being

Input values: $l = 6\text{m}$, $b = 20\text{mm}$, $h = 80\text{mm}$, $f_{yk} = 235\text{MPa}$, $\gamma_{M0} = 1.15$

Solution:

\[
M_{Ed,pl} = \frac{1}{8}q_{d,pl}l^2
\]

\[
W_{y,pl} = \frac{1}{4}bh^2 = 3.2 \times 10^{-5}\text{m}^3
\]

\[
f_{yd} = \frac{f_{yk}}{\gamma_{M0}} = 204.35\text{MPa}
\]

\[
f_{yd} = \frac{M_{sd,pl}}{W_{y,pl}} = \frac{q_{d,pl}l^2}{8W_{y,pl}}
\]

\[
\Rightarrow q_{d,pl} = \frac{8f_{yd}W_{y,pl}}{l^2} = 1.45\text{kN/m}
\]
Example 2 - summary

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

\[ \varepsilon_x \]

\[ \sigma_x \]

\[ \varepsilon_x \]

Stress-strain diagram

\[ f_y \]

\[ Y \]

\[ M_{Rd,pl} \]

\[ M_{Rd,el} \]

\[ f_{yd} \]

\[ -f_{yd} \]

Plastized state

Plastic reserve of rectangular cross-section

\[ q_{d,pl} = \frac{1,45}{0,97} = 1,5 \]

\[ \Rightarrow 50\% \]

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<thead>
<tr>
<th>state</th>
<th>( W_y ) [m(^3)]</th>
<th>( q_d ) [kN/m]</th>
</tr>
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<tbody>
<tr>
<td>a)</td>
<td>( 2,13 \times 10^{-5} )</td>
<td>0,97</td>
</tr>
<tr>
<td>b)</td>
<td>( 2,93 \times 10^{-5} )</td>
<td>1,33</td>
</tr>
<tr>
<td>c)</td>
<td>( 3,20 \times 10^{-5} )</td>
<td>1,45</td>
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