Elastic-plastic stress-strain analysis of prismatic bar under bending
### Ideal elastic-plastic material

<table>
<thead>
<tr>
<th>section</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y-Y'</td>
<td>Hook’s law</td>
</tr>
<tr>
<td>Y-A</td>
<td>Plastic state – free increase of deformations (strain)</td>
</tr>
<tr>
<td>A-B</td>
<td>Unloading</td>
</tr>
<tr>
<td>B-C</td>
<td>Re-increasing of strain</td>
</tr>
</tbody>
</table>

#### Stress-strain diagram

- **Tension**
  - Hook’s law: $\sigma = f_y$ for $\epsilon > 0$
  - Plastic state: $\alpha = \arctan E$
  - Re-increasing of strain: $\epsilon_p$ ...

- **Compression**
  - Unloading: $\epsilon_e$ ...
  - Plastic (permanent) deformation: $\epsilon_p$ ...
  - Elastic deformation: $\epsilon_e$ ...

**Elasto-plastic stress analysis of prismatic bar under bending**
Elastic stress-strain analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

σₓ stress behavior

σₓ,max

compression

tension

σₓ,max = \( \frac{M_{Ed}}{W_y} \)

fᵧd

MEd ≤ Mrd,el = Wᵧ.fᵧd = \( \frac{1}{6} \) b.h².fᵧd

|σₓ,max| < fᵧd

σₓ Stress-strain diagram

outer fibres of cross-section

εₓ

state I.

Bending (direct) stress in outer fibres

fᵧd = \( \frac{f_{yk}}{γ_M} \)
Elastic stress-strain analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

σ<sub>x</sub> stress behavior

Stress-strain diagram

state II.

Bending (direct) stress in outer fibres

\[ \sigma_{x,max} = f_{yd} \]

\[ M_{Rd,el} = W \cdot f_{yd} = \frac{1}{6} b h^2 f_{yd} \]

Elasto-plastic stress analysis of prismatic bar under bending
Load increases permanently, plastic zones come into being: 1 – under tension, 2 – under compression. Cross-section residuum has still elastic behavior.

Bernoulli hypothesis is still valid, $\varepsilon_x$ is continuously linear. In A point is

\[ \varepsilon_x = \frac{f_{yd}}{E} \]
Elastic-plastic stress-strain analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

σₓ stress behavior

σₓ Stress-strain diagram

Cross-section is plastized, so-called plastic joint comes into being, carrying capacity under bending is exhausted.

state IV.
Elastic-plastic stress-strain analysis

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

Cross-section

\( A_1 \) \\
\( y \) \\
\( A_2 \) \\
\( z \) \\
\( b \) \\
\( h \)

\( \sigma_x \) stress behavior

- Compression
- Tension

\( f_{yd} \)

\( M_{Rd,pl} \)

Stress-strain diagram

\( \sigma_x \)

\( f_y \)

\( Y \)

Plastic joint

Specifically:

State IV.

\( W_{y,pl} = \frac{1}{4}bh^2 \)

Plastic reserve of rectangular cross-section

\[
\frac{W_{y,pl}}{W_{y,el}} = \frac{\frac{1}{4}bh^2}{\frac{1}{6}bh^2} = \frac{6}{4} = 1.5 \quad \rightarrow \quad 50\%
\]

Elasto-plastic stress analysis of prismatic bar under bending
Example 1 – Section modulus during plasticisation of the section

Determine modulus of the section of the rectangular cross-section b x h for this states:
1. Yield limit $f_y$ is reached in the edge of the section.
2. Outer quarters of the section are plasticised.
3. The whole section is plasticised.

Stress-strain diagram

$\sigma_x$ vs $\varepsilon_x$

$\sigma_x = f_y$ for $\varepsilon_x = 0$

1. $f_y$ to the right
2. $f_y$ to the left
3. $f_y$ to the right

Plasticised zone

\[ M_{Rd,el} \]
\[ M_{Rd,el,pl} \]
Example 1 – Section modulus during plasticisation of the section

Determine modulus of the section of the rectangular cross-section b x h for this state:

1. Yield limit \( f_y \) is reached in the edge of the section.

\[
M_{y1} = F \cdot r = \frac{1}{2} \cdot f_y \cdot \frac{h}{2} \cdot b \cdot \frac{2}{3} h = \frac{1}{6} b h^2 f_y
\]

\[
M_{y1} = f_y \cdot W_{y1} \Rightarrow W_{y1} = \frac{M_{y1}}{f_y} = \frac{1}{6} b h^2 = W_{y,el}
\]
Example 1 – Section modulus during plasticisation of the section

Determine modulus of the section of the rectangular cross-section $b \times h$ for this state:

1. Outer quarters of the section are plasticised.

\[ M_{y2} = F_1 \cdot r_1 + F_2 \cdot r_2 = f_y \cdot \frac{h}{4} \cdot b \cdot \frac{3}{4} h + \frac{1}{2} \cdot f_y \cdot \frac{h}{4} \cdot b \cdot \frac{1}{3} h = \left( \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{8} \cdot \frac{1}{3} \right) bh^2 f_y = \frac{11}{48} \cdot bh^2 f_y \]

\[ W_{y2} = \frac{M_{y2}}{f_y} = \frac{11}{48} bh^2 \Rightarrow (= 1.375 \cdot W_{y,el}) \]
Example 1 – Section modulus during plasticisation of the section

Determine modulus of the section of the rectangular cross-section b x h for this state:
3. The whole section is plasticised.

\[ M_{y3} = F \cdot r = f_y \cdot \frac{h}{2} \cdot b \cdot \frac{h}{2} = \frac{1}{4} bh^2 f_y \]

\[ W_{y3} = \frac{M_{y3}}{f_y} = \frac{1}{4} bh^2 \Rightarrow (= 1.5 \cdot W_{y,el}) \]
Example 1 – Section modulus during plasticisation of the section

Determine modulus of the section for this state:

1. Yield limit \( f_y \) is reached in the edge of the section.
2. Outer quarters of the section are plasticised.
3. The whole section is plasticised.

<table>
<thead>
<tr>
<th>state</th>
<th>( W_y [m^3] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \frac{1}{6} b.h^2 )</td>
</tr>
<tr>
<td>2.</td>
<td>( \frac{11}{48} b.h^2 )</td>
</tr>
<tr>
<td>3.</td>
<td>( \frac{1}{4} b.h^2 )</td>
</tr>
</tbody>
</table>
Example 2 – Elastic-plastic behaviour of material

Determine \( q_{\text{max}} \) for the following states:

1. \( \sigma_x \) reaches yield limit at the edges of the section = elastic state
2. Outer quarters of the section are plasticized = elastic-plastic state
3. The whole section is plasticized

\( (l = 6m, b = 2cm, h = 10cm, \text{Fe 360/S235}, \gamma_{M0} = 1,00) \)

\[
\sigma_x = \frac{M_{Ed}}{W_y} \leq f_{yd}
\]

\[
M_{Ed} = \frac{1}{8} q l^2
\]

\[
q_{\text{max}} = ?
\]

\[
\begin{align*}
\text{state} & \quad W_y [\text{m}^3] \\
1. & \quad \frac{1}{6} . b . h^2 \\
2. & \quad \frac{11}{48} . b . h^2 \\
3. & \quad \frac{1}{4} . b . h^2 \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>state</th>
<th>( W_y [\text{m}^3] )</th>
<th>( q_d [\text{kN/m}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( 3,333 \times 10^{-5} )</td>
<td>( 1,74 )</td>
</tr>
<tr>
<td>2.</td>
<td>( 4,583 \times 10^{-5} )</td>
<td>( 2,39 )</td>
</tr>
<tr>
<td>3.</td>
<td>( 5,0 \times 10^{-5} )</td>
<td>( 2,61 )</td>
</tr>
</tbody>
</table>
Example 2 - summary

Ideal elasto-plastic material, prismatic bar, rectangular cross-section

1. 

2. 

3. 

Plastized

Plastic reserve of rectangular cross-section

\[
\frac{q_{d,pl}}{q_{d,el}} = \frac{1,45}{0,97} = 1,5 \rightarrow 50\%
\]

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