Analysis of Statically Indeterminate Structures by the Displacement Method
Methods of structural analysis

The methods are classified into two groups:

1. **Force method of analysis** (Statics of building structures I)
   - Primary unknowns are forces and moments
   - Deformation conditions are written depending on pre-selected statically indeterminate reactions.
   - The unknown statically indeterminate reactions are evaluated solving these equations.
   - The remaining reactions are obtained from the equilibrium equations.
Methods of structural analysis

- The methods are classified into two groups:
  2. **Displacement method of analysis**
    - Primary unknowns are displacements.
    - Equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load-displacement relations.
    - Unknown joint displacements are calculated by solving equilibrium equations.
    - In the next step, the unknown reactions are computed from compatibility equations using force displacement relations.
Displacement method

- This method follows essentially the same steps for both statically determinate and indeterminate structures.
- Once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis (hence, this method is preferred to computer implementation).
Displacement method

1. Slope-Deflection Method
   - In this method it is assumed that all deformations are due to bending only. Deformations due to axial forces are neglected.

2. Direct Stiffness Method
   - Deformations due to axial forces are not neglected.

The Slope-deflection method was used for many years before the computer era. After the revolution occurred in the field of computing direct stiffness method is preferred.
Slope-Deflection Method: Beams
Slope-Deflection Method: Beams

- Application of Slope-Deflection Equations to **Statically Indeterminate Beams**:
  - The procedure is the same whether it is applied to beams or frames.
  - It may be summarized as follows:
    1. Identify all kinematic degrees of freedom for the given problem. Degrees of freedom are treated as unknowns in slope-deflection method.
    2. Determine the fixed end moments at each end of the span to applied load (using table).
    3. Express all internal end moments by slope-deflection equations in terms of:
       - fixed end moments
       - near end and far end joint rotations
Slope-Deflection Method: Beams

4. Write down one equilibrium equation for each unknown joint rotation. Write down as many equilibrium equations as there are unknown joint rotations. Solve the set of equilibrium equations for joint rotations.

5. Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.

6. Evaluate shear forces and reactions.

7. Draw bending moment and shear force diagrams.
Slope-Deflection Method: Beams

Example 1

1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to first degree. Only one joint rotation $\phi_b$ is unknown.
2. Fixed end moments are calculated referring to the table.

\[
\begin{align*}
\overline{M}_{ab} &= -\frac{1}{8} \cdot F \cdot l_{ab} - \frac{1}{12} \cdot q_1 \cdot l_{ab}^2 \\
\overline{M}_{ba} &= +\frac{1}{8} \cdot F \cdot l_{ab} + \frac{1}{12} \cdot q_1 \cdot l_{ab}^2 \\
\overline{M}_{bc} &= -\frac{1}{12} \cdot q_2 \cdot l_{bc} \\
\overline{M}_{cb} &= +\frac{1}{12} \cdot q_2 \cdot l_{bc}^2
\end{align*}
\]
3. Express internal end moments by slope-deflection equations.

\[ M_{ab} = \bar{M}_{ab} + \frac{2EI}{l_{ab}}(2 \cdot \varphi_a + \varphi_b) \]

\[ M_{ba} = \bar{M}_{ba} + \frac{2EI}{l_{ab}}(2 \cdot \varphi_b + \varphi_a) \]

\[ M_{bc} = \bar{M}_{bc} + \frac{2EI}{l_{bc}}(2 \cdot \varphi_b + \varphi_c) \]

\[ M_{cb} = \bar{M}_{cb} + \frac{2EI}{l_{bc}}(2 \cdot \varphi_c + \varphi_b) \]
4. Equilibrium equations (write one equilibrium equation for each unknown joint rotation)

- End moments are expressed in terms of unknown rotation $\varphi_b$. Now, the required equation to solve for the rotation $\varphi_b$ is the moment equilibrium equation at support $b$.

$$\sum M_b = 0: M_{ba} + M_{bc} = 0 \Rightarrow \varphi_b$$

5. End moments

- After evaluating $\varphi_b$, substitute it to evaluate beam end moments.
6. **Shear forces** and reactions

- Now, reactions at supports are evaluated using equilibrium equations. Shear forces are equal to plus/minus this reactions.

\[
V_{ab} = R_{ab} = \frac{1}{l_{ab}} \left( F \cdot \frac{l_{ab}}{2} + q_1 \cdot \frac{l_{ab}^2}{2} - M_{ab} - M_{ba} \right)
\]

\[
V_{ba} = -R_{ba} = -\frac{1}{l_{ab}} \left( F \cdot \frac{l_{ab}}{2} + q_1 \cdot \frac{l_{ab}^2}{2} + M_{ab} + M_{ba} \right)
\]

\[
V_{bc} = R_{bc} = \frac{1}{l_{bc}} \left( q_2 \cdot \frac{l_{bc}^2}{2} - M_{bc} - M_{cb} \right)
\]

\[
V_{cb} = -R_{cb} = -\frac{1}{l_{bc}} \left( q_2 \cdot \frac{l_{bc}^2}{2} + M_{bc} + M_{cb} \right)
\]
6. Shear forces and **reactions**

- Now, reactions at supports are evaluated using equilibrium equations. Shear forces are equal to plus/minus this reactions.

\[
R_{az} = R_{ab} \\
R_{bz} = R_{ba} + R_{bc} \\
R_{cz} = R_{cb}
\]
7. Draw shear force and bending moment diagrams.
Slope-Deflection Method: Beams

Example 1

\[ M_{bc} = -M_{ba} \]

\[ M_{ab} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} M_{ab} \\ M_{bc} \end{pmatrix} + \begin{pmatrix} R_{ab} \\ R_{bc} \end{pmatrix} \begin{pmatrix} l_{ab} \\ l_{bc} \end{pmatrix} - q_1 \begin{pmatrix} \left( \frac{l_{ab}}{2} \right)^2 \\ \frac{x_n^2}{2} \end{pmatrix} \cdot \frac{1}{2} - q_2 \begin{pmatrix} \frac{l_{ab}}{2} \end{pmatrix} \cdot \frac{1}{2} \]

\[ M_{max} = M_{bc} + R_{bc} \cdot x_n - q_2 \cdot \frac{x_n^2}{2} \]

\[ F = 20 \text{ kN} \]

\[ q_1 = 2 \text{ kN/m} \]

\[ q_2 = 4 \text{ kN/m} \]

\[ EI = \text{konst.} \]

\[ l_{ab}, l_{bc}, x_n \]
1. Degrees of freedom

- The continuous beam is kinematically indeterminate to second degree.

  - The 1st possibility of solution - two unknown joint rotation $\varphi_b$, $\varphi_c$ ($\varphi_a = 0$) - two required equations to solve for the rotation $\varphi_b$, $\varphi_c$ are the moment equilibrium equations at support $b$ and $c$. 

$$q = 3 \text{ kN/m}$$

$$F_1 = 10 \text{ kN}$$

$$F_2 = 5 \text{ kN}$$

$$EI_{ab} = EI$$

$$EI_{bc} = 2EI$$
2. Fixed end moments are calculated referring to the table.

\[
\overline{M}_{ab} = -\frac{1}{12} \cdot q \cdot l_{ab}^2 \\
\overline{M}_{ba} = +\frac{1}{12} \cdot q \cdot l_{ab}^2 \\
\overline{M}_{bc} = -\frac{1}{8} \cdot F \cdot l_{bc} \\
\overline{M}_{cb} = +\frac{1}{8} \cdot F \cdot l_{bc}
\]
3. Express internal end moments by slope-deflection equations.

\[ M_{ab} = \overline{M}_{ab} + \frac{2EI_{ab}}{l_{ab}}(2 \cdot \varphi_a + \varphi_b) \]

\[ M_{ba} = \overline{M}_{ba} + \frac{2EI_{ab}}{l_{ab}}(2 \cdot \varphi_b + \varphi_a) \]

\[ M_{bc} = \overline{M}_{bc} + \frac{2EI_{bc}}{l_{bc}}(2 \cdot \varphi_b + \varphi_c) \]

\[ M_{cb} = \overline{M}_{cb} + \frac{2EI_{bc}}{l_{bc}}(2 \cdot \varphi_c + \varphi_b) \]

\[ M_{cd} = -F_2 \cdot 3 \]
4. Equilibrium equations (write two equilibrium equations for two unknown joint rotations)

- End moments are expressed in terms of unknown rotations. Now, the required equations to solve for the rotations are the moment equilibrium equations at supports $b$ and $c$.

\[
\begin{align*}
\sum M_b &= 0: \quad M_{ba} + M_{bc} = 0 \\
\sum M_c &= 0: \quad M_{cb} + M_{cd} = 0
\end{align*}
\]

$\Rightarrow \varphi_b, \varphi_c$

5. End moments

- After evaluating $\varphi_b, \varphi_c$, substitute them to evaluate end moments.

Then the procedure is the same as for the 2\textsuperscript{nd} possibility of solution.
Slope-Deflection Method: Beams

Example 2, the 2\textsuperscript{nd} possibility of solution

1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to second degree.

   The 2\textsuperscript{nd} possibility of solution - solve only one unknown joint rotation $\varphi_b$ ($\varphi_a = 0$, joint rotation $\varphi_c$ is not necessary to solution because the moment in the cantilever portion $M_c$ is known $\Rightarrow$ beam portion $bc$ is taken as fixed - hinged).
Slope-Deflection Method: Beams

Example 2, the 2\textsuperscript{nd} possibility of solution

2. Fixed end moments are calculated referring to the table.

\[
\overline{M}_{ab} = -\frac{1}{12} \cdot q \cdot l_{ab}^2
\]

\[
\overline{M}_{ba} = +\frac{1}{12} \cdot q \cdot l_{ab}^2
\]

\[
\overline{M}_{bc} = -\frac{3}{16} \cdot F \cdot l_{bc} + \frac{M_c}{2}
\]

\[
F_1 = 10 \text{kN}
\]

\[
M_c = F_2 \cdot 3 = 15 \text{kNm}
\]
3. Express internal end moments by slope-deflection equations.

\[ M_{ab} = \overline{M}_{ab} + \frac{2EI_{ab}}{l_{ab}} (2 \cdot \varphi_a + \varphi_b) \]

\[ M_{ba} = \overline{M}_{ba} + \frac{2EI_{ab}}{l_{ab}} (2 \cdot \varphi_b + \varphi_a) \]

\[ M_{bc} = \overline{M}_{bc} + \frac{3EI_{bc}}{l_{bc}} \cdot \varphi_b \]
4. Equilibrium equations (write one equilibrium equation for each unknown joint rotation)
   - End moments are expressed in terms of unknown rotation $\phi_b$. Now, the required equation to solve for the rotation $\phi_b$ is the moment equilibrium equation at support $b$.

   $\sum M_b = 0 : M_{ba} + M_{bc} = 0 \Rightarrow \phi_b$

5. End moments
   - After evaluating $\phi_b$, substitute it to evaluate beam end moments.

Then the procedure is the same as for the 1st possibility of solution.
6. **Shear forces** and reactions.

\[ V_{ab} = R_{ab} = \frac{1}{l_{ab}} \left( q_1 \cdot \frac{l_{ab}^2}{2} - M_{ab} - M_{ba} \right) \]

\[ V_{ba} = -R_{ba} = -\frac{1}{l_{ab}} \left( q_1 \cdot \frac{l_{ab}^2}{2} + M_{ab} + M_{ba} \right) \]

\[ V_{bc} = R_{bc} = \frac{1}{l_{bc}} \left( F_1 \cdot \frac{l_{bc}}{2} - M_{bc} - M_{cb} \right) \]

\[ V_{cb} = -R_{cb} = -\frac{1}{l_{bc}} \left( F_1 \cdot \frac{l_{bc}}{2} + M_{bc} + M_{cb} \right) \]
6. Shear forces and reactions.

\[ F_1 = 10 \text{ kN} \]

\[
\begin{align*}
R_{az} &= R_{ab} \\
R_{bz} &= R_{ba} + R_{bc} \\
R_{cz} &= R_{cb} + F_2
\end{align*}
\]
7. Draw shear force and bending moment diagrams.
Slope-Deflection Method: Beams

Example 2

\[ M_{\text{max}} = M_{ab} + R_{ab} \cdot x_n - q \cdot \frac{x_n^2}{2} \]

\[ M_F = M_{bc} + R_{bc} \cdot \frac{l_{bc}}{2} \]

\[ q = 3 \text{ kN/m} \]

\[ F_1 = 10 \text{ kN} \]

\[ F_2 = 5 \text{ kN} \]
Slope-Deflection Method: Beams

Example 3

\[ q_1 = 2 \text{kN/m} \]

\[ q_2 = 3 \text{kN/m} \]

\[ F = 6 \text{kN} \]

\[ F = 6 \text{kN} \]
Slope-Deflection Method: Beams

Example 3

\[ x_m = \sqrt{\frac{1V_{c61} \cdot 2 \cdot b \cdot c}{q_2}} = \sqrt{\frac{4.31 \cdot 2 \cdot 4}{3}} = 3.39 \text{m} \]
Slope-Deflection Method: Beams

Example 3

\[ M_{\text{max}} = M_{cb} + R_{cb} \cdot x_n - \frac{q_2 \cdot x_n^3}{6 \cdot L_{bc}} \]

\[ M_{\text{max}} = -7.77 \cdot 4 + 4.31 \cdot 3.39 - \frac{3.39^3}{6.4} = 1.97 \text{ kNm} \]

\[ M_{F1} = M_{cd} + R_{cd} \cdot 2 = -7.77 + 7.29 \cdot 2 = 6.81 \text{ kNm} \]

\[ M_{F2} = M_{cd} + R_{cd} \cdot 4 - F \cdot 2 = -7.77 + 7.29 \cdot 4 - 6 \cdot 2 = 9.39 \text{ kNm} \]