Non-linear mechanics

- types of non-linearities in structural mechanics,

- introduction to the theory of plasticity,

- solution methods for non-linear problems.
Types of non-linearities

- construction (model) non-linearities – supports and structural elements that are working only in certain conditions (compression-only etc.),

- physical (constitutive) non-linearities – material behaviour is not linear (differs from Hooke’s law) (non-linear elasticity, plasticity, fracture mechanics, . . . ),

- geometric non-linearity – large deformations (displacements, rotations, . . . )
Model non-linearity

- supports that are working only under certain type of load,
- often used for contact problems,
- usually requires iterative solution.
Support working only for certain load (1)
Support working only for certain load (2)

Model:
Support working only for certain load (3)

Reactions (normal supports):

Results 23. 09. 2008

-7.135765e+02
-6.243795e+02
-5.351824e+02
-4.459853e+02
-3.567883e+02
-2.675912e+02
-1.783941e+02
-8.919706e+01
0.000000e+00
5.488281e+01
1.097656e+02
1.646484e+02
2.195312e+02
2.744141e+02
3.292969e+02
3.841797e+02
4.390625e+02

23. 09. 2008

podložka
Support working only for certain load (4)

Deformed shape (normal supports):
Support working only for certain load (5)

Normal stress $\sigma_x$ (normal supports):
Support working only for certain load (6)

Normal stress $\sigma_y$ (normal supports):

![Diagram showing stress distribution](image.png)
Support working only for certain load (7)

Normal stress $\sigma_1$ (normal supports):

23. 09. 2008
Support working only for certain load (8)

Deformed shape (compression-only supports):
Support working only for certain load (9)

Normal stress $\sigma_x$ (compression-only supports):
Support working only for certain load (10)

Normal stress $\sigma_y$ (compression-only supports):
Support working only for certain load (11)

Normal stress $\sigma_1$ (compression-only supports):
Physical (constitutive) non-linearity

- Non-linear elasticity
  - Hooke’s law is not respected
  - No non-reversible deformations
- Elasto-plastic behaviour
  - No-reversible (plastic) deformations
- Viskoelasticity, viskoplasticty, . . .
- Fragile materials
  - Fracture mechanics
Elastoplastic behaviour (1)

Load-displacement diagram for axially loaded member:
Elastoplastic behaviour (2)

1. Ideally elasto–plastic

2. Elasto–plastic with hardening

3. Stiff–plastic
Plasticity condition

\[ f() \]

\[ \sigma_1 \quad \sigma_2 \]

\[ u \]

\[ F \]

\[ f() \]
Plasticity conditions (1)

Maximal normal stresses theory (Rankine):

\[-\sigma_{md} \leq \sigma_1 \leq \sigma_{mt}\]

\[\sigma_1 - \sigma_{mt} = 0\]

\[\sigma_2 - \sigma_{md} = 0\]
Plasticity conditions (2)

Maximal shear stresses theory (Tresca):

\[ \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} - \tau_m = 0 \]

\[ \sigma_1 - \sigma_3 - \sigma_{mt} = 0 \]

\[ (\tau_m = \frac{\sigma_{mt}}{2}) \]

\[ \sigma_{md} = \sigma_{mt} \]
Plasticity conditions (3)

Shape change energy condition (von Mises) (von Mises, Huber, Hencky):

\[(\sigma_1 - \sigma_2)^2 + (\sigma_3 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_{mt}^2\]

\[\sigma_{md} = \sigma_{mt}\]

Note: „von Mises stress“:

\[\sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_3 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2}{2}}\]
Plasticity conditions (4)

Mohr – Coulomb condition (for geotechnics):

\[
\sigma_1 - \frac{\sigma_{mt}}{\sigma_{md}} \sigma_3 - \sigma_{mt} = 0
\]

\[\sigma_{md} \neq \sigma_{mt}\]
Plasticity conditions (4)

Chen – Chen condition (concrete):

For compression–compression area \((\sigma_1 < 0 \text{ and } \sigma_2 < 0, \sigma_3 < 0)\):

\[
J_2 + \frac{A_{yc}}{3} I_1 - \tau_{yc}^2 = 0
\]

For other areas:

\[
J_2 - \frac{1}{6} I_1^2 + \frac{A_{yt}}{3} I_1 - \tau_{yt}^2 = 0
\]
Hardening (1)
Hardening (2)

- **Kinematic**
  - subsequent plasticity conditions are moving
  - no change of shape and size

- **Isotropic**
  - subsequent conditions are changing size proportionally
  - no moves

- **Combined**
  - kinematic + isotropic
Hardening (3)

Combined hardening
Plasticity and failure conditions

1. Initial plasticity condition
2. Subsequent plasticity condition
3. Failure condition
Variants of theory of plasticity (1)

Theory of plastic deformations:

- uses relations between total deformations and stresses:
  \[ \sigma = D^{EP} \varepsilon \]
- solution **does not depend** on loading path
- (in most cases) hard derivation of relations
Variants of theory of plasticity (2)

Plastic flow theory:

- relation between changes ("speeds") of deformations and stresses:

\[ \dot{\sigma} = D^{ep} \dot{\epsilon} \]

- solution depends on loading path
- solution can be divided into a set of linearized steps
Plastic flow theory

Unkowns – changes ("speeds"):

- stresses: \( \dot{\sigma} = \{ \dot{\sigma}_x, \dot{\sigma}_y, \dot{\sigma}_z, \dot{\tau}_{yz}, \dot{\tau}_{yz}, \dot{\tau}_{xy} \}^T \)
- relative deformations: \( \dot{\epsilon} = \{ \dot{\epsilon}_x, \dot{\epsilon}_y, \dot{\epsilon}_z, \dot{\gamma}_{yz}, \dot{\gamma}_{yz}, \dot{\gamma}_{xy} \}^T \)
- displacements (and rotations): \( \dot{u} = \{ \dot{u}, \dot{v}, \dot{w} \} \)

Assumptions:

- initial stress \( \sigma \) and strain \( \epsilon \) state must be known
- solution have to respect boundary conditions
Elasto–plastic material matrix (1)

- Constitutive equations:

\[ \dot{\sigma} = D^{ep} \dot{\varepsilon} \]

\( D^{ep} \) ... elasto–plastic material matrix (have to be found)

- Division of change of deformations to elastic and plastic part:

\[ \dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p \]

- Plastic condition (used for description of change from elastic to plastic state):

\[ f(\sigma, k) = 0 \]
Elasto–plastic material matrix (2)

- Consistence condition of plastic material:

\[ df = \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \{ d\sigma \} + \left\{ \frac{\partial f}{\partial k} \right\}^T \{ dk \} = 0 \]
Elasto–plastic material matrix (3)

• Speed of plastic deformation (plastic deformation law):

\[ \dot{\varepsilon}_p = d\lambda \left\{ \frac{\partial f}{\partial \sigma} \right\} \]

• Stress changes:

\[ \dot{\sigma} = d\sigma = D_e (\dot{\varepsilon} - \dot{\varepsilon}_p) = D_e \left( \dot{\varepsilon} - d\lambda \left\{ \frac{\partial f}{\partial \sigma} \right\} \right) \]

• Equivalent plastic deformation:

\[ d\varepsilon_p = \sqrt{\dot{\varepsilon}_p^T \dot{\varepsilon}_p} = d\lambda \sqrt{\left\{ \frac{\partial f}{\partial \sigma} \right\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\} } \]
Elasto–plastic material matrix (4)

- From consistence condition:

\[
\begin{align*}
\left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e d\mathbf{\varepsilon} - d\lambda \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e \left\{ \frac{\partial f}{\partial \sigma} \right\} + d\lambda \frac{\partial f}{\partial \varepsilon_p} \sqrt{\left\{ \frac{\partial f}{\partial \sigma} \right\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\}} = 0
\end{align*}
\]
Elasto–plastic material matrix (3)

Computation of \( d\lambda \) parameter:

\[
\begin{align*}
d\lambda &= \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e \dot{\varepsilon} \\
&= \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e \left\{ \frac{\partial f}{\partial \sigma} \right\} + \frac{\partial f}{\partial \varepsilon_p} \sqrt{\left\{ \frac{\partial f}{\partial \sigma} \right\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\}} \\
&= \frac{\partial f}{\partial \sigma} \left( \dot{\varepsilon} - d\lambda \left\{ \frac{\partial f}{\partial \sigma} \right\} \right)
\end{align*}
\]

If put to \( \dot{\sigma} = \mathbf{D}_e \left( \dot{\varepsilon} - d\lambda \left\{ \frac{\partial f}{\partial \sigma} \right\} \right) \):

\[
\begin{align*}
\dot{\sigma} &= \mathbf{D}_e \left( \dot{\varepsilon} - \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e \dot{\varepsilon} \right) \\
&= \mathbf{D}_e \left( \dot{\varepsilon} - \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e \dot{\varepsilon} \right) \\
&= \mathbf{D}_e \left( \dot{\varepsilon} - \left\{ \frac{\partial f}{\partial \sigma} \right\}^T \mathbf{D}_e \dot{\varepsilon} \right)
\end{align*}
\]
Elasto–plastic material matrix (4)

The equation for $\dot{\sigma}$ can be simplified:

$$\dot{\sigma} = D_{ep} \dot{\varepsilon}_{ep},$$

where elasto–plastic material matrix $D_{ep}$ is:

$$D_{ep} = D_e - \frac{D_e \left\{ \frac{\partial f}{\partial \sigma} \right\} \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e}{\left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e \left\{ \frac{\partial f}{\partial \sigma} \right\} - \frac{\partial f}{\partial \varepsilon_p} \sqrt{\left\{ \frac{\partial f}{\partial \sigma} \right\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\}}}$$
Material models for concrete

Mechanical properties of concrete

- Near no linear elastic behaviour
- Non-linear behaviour (non-reversible deformations, cracking, …)
- Different behaviour for different types of loading
Constitutive models for concrete

- **Discrete models:**
  individual cracks are modelled (usually by finite element mesh changes)

- **Continuum models:**
  model is assumed to remain continuous but with different material properties
  - Models based on non-linear fracture mechanics (smeared cracks, non-local continuum, microplane models)
  - **Elasto–plastic models**
Chen plasticity condition (1)

- Usual plasticity conditions don’t satisfy concrete behaviour (von Mises, Tresca)
- Experiment research by plane stress concrete samples (Kupfer)
- Several approximations of experimental data (Kupfer, Chen a Chen, Willam a Warnke, ...)

Chen and Chen:

- Approximation of Kupfer data by polynomic functions
- The condition can be used both for plasticity and for failure
Chen plasticity condition (2)

For compression–compression zone 
\((\sigma_1 < 0 \text{ a } \sigma_2 < 0, \sigma_3 < 0)\):

\[ J_2 + \frac{A_{yc}}{3} I_1 - \tau_{yc}^2 = 0 \]

For all other zones:

\[ J_2 - \frac{1}{6} I_2^2 + \frac{A_{yt}}{3} I_1 - \tau_{yt}^2 = 0 \]

where:

\[ I_1 = \sigma_1 + \sigma_2 + \sigma_3 \]

\[ J_2 = \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \]
Chen plasticity condition (3)

Constants $A_{yx}$, $\tau_{yx}$ explanation:

$$A_{yc} = \frac{f_{ybc}^2 - f_{yc}^2}{2f_{ybc} - f_{yc}}$$

$$\tau_{yc}^2 = \frac{f_{ybc}f_{yc}(2f_{yc} - f_{ybc})}{3(2f_{ybc} - f_{yc})}$$

$$A_{yt} = \frac{f_{yc} - f_{yt}}{2f_{yc}f_{yt}}$$

$$\tau_{ut}^2 = \frac{2}{6}$$
Chen failure condition (4)

Failure condition can be defined in the same manner:

\[ J_2 + \frac{A_{uc}}{3} I_1 - \tau_{uc}^2 = 0 \]

\[ J_2 - \frac{1}{6} I_1^2 + \frac{A_{ut}}{3} I_1 - \tau_{ut}^2 = 0 \]

\[ A_{uc} = \frac{f_{ubc}^2 - f_{uc}^2}{2f_{ubc} - f_{uc}} \]

\[ \tau_{yc}^2 = \frac{f_{ubc}f_{yc}(2f_{uc} - f_{ubc})}{3(2f_{ubc} - f_{uc})} \]

\[ A_{ut} = \frac{f_{uc} - f_{ut}}{f_{uc}f_{ut}} \]

\[ \tau_{ut}^2 = \frac{2}{6} \]
Chen failure condition (5)

Intermediate (subsequent) conditions:

\[ A_c = \alpha_c \tau_c^2 + \beta_c \]
\[ A_t = \alpha_t \tau_t^2 + \beta_t \]

\[ \alpha_c = \frac{A_{uc} - A_{yc}}{\tau_{uc}^2 - \tau_{yc}^2} \]
\[ \beta_c = \frac{A_{yc} \tau_{uc}^2 - A_{yc} \tau_{yc}^2}{\tau_{uc}^2 - \tau_{yc}^2} \]
\[ \alpha_t = \frac{A_{ut} - A_{yt}}{\tau_{ut}^2 - \tau_{yt}^2} \]
\[ \beta_t = \frac{A_{yt} \tau_{ut}^2 - A_{yt} \tau_{yt}^2}{\tau_{ut}^2 - \tau_{yt}^2} \]
Related conditions

Kupfer failure condition:
- defined for 2D stress state
- uses data from standardized tests (cylindric strength of concrete)

Willam–Warnke condition:
- defined for 3D
- very similar to Chen one in term of input data and shape:

\[ f = \frac{1}{3z\sigma_c} I_1 + \sqrt{\frac{2}{5}} + \frac{1}{r(\theta)\sigma_c} J_2 - 1 = 0 \]
Example – finite element model of concrete arch
Example – plastic areas on arch
Example – stress–strain curve
Smeared crack model (1)

- Modelling of damaged area (cracks, ...) by reduction of material properties \((E, \nu)\)
- Continuous model
- Not very good for large discrete cracks
Smeared crack model (2)

Orthotropic material:

\[ D = \frac{R_2}{R_2 - \mu^2 R_1} \begin{bmatrix} R_1 & \mu R_1 & 0 & 0 \\ \mu R_1 & R_2 & 0 & \frac{\beta G}{R_2/(R_2 - \mu^2 R_1)} \\ 0 & 0 & R_2/(R_2 - \mu^2 R_1) & 0 \end{bmatrix} \]
Smeared crack model (3)

One-dimensional equivalent stress–strain relation:

- 2D failure condition is used for determination of equivalent diagram parameters:
  - Chen–Chen
  - Kupfer
- Crack band model can be used for limiting of result dependence of finite element mesh size (fracture energy is used)
Smeared crack model (4)

Bažant’s crack band model:

Fracture energy:

\[ G_F = A_G L = \text{const.}, \]

\[ G_F = \int_0^\infty \sigma_n(w) \, dw, \]

Total cracks width: \( w = \varepsilon L \)

Descending modulus form:

\[ E_z = \frac{E_o}{1 - \frac{2G_F E_o}{L \sigma_{max}^2}}. \]
Smeared crack model (5)

Example – finite element model:
Smeared crack model (6)

Example – residual stiffness $R_t$: 
Smeared crack model (7)

Example – load–displacement curve:
Recomended reading

- OHTANI, Y., CHEN, W. F. Multiple Hardening Plasticity for Concrete Materials, Journal of the EDM ASCE, 1988