ADVANCED STRUCTURAL MECHANICS

Lecture 1

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Topics

1. Basic equations
2. Plane structures
3. Energetical principles, variational methods
4. Finite element method
Recommended books


Topic 1: Basic equations of elasticity

- Basic unknowns
- Geometrical relations
- Equilibrium equations
- Material law
- Compatibility equations
Basic unknowns

Stress vector

\[ \boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}^T \]  

(1)

Strain vector

\[ \boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\}^T \]  

(2)

Displacement vector

\[ \mathbf{u} = \{u_x, u_y, u_z\}^T \]  

(3)
Geometrical relations (1)
Geometrical relations (2)

\[ \varepsilon_x = \frac{A'B' - AB}{AB} = \frac{(x + dx + u + \frac{\partial u}{\partial x} dx) - (x + u) - dx}{dx} = \frac{\partial u}{\partial x} \]
Geometrical relations (3)

Normal deformations

\[ \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad (4) \]

Shear deformations

\[ \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad (5) \]
\[ \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}, \quad (6) \]
\[ \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \quad (7) \]
Geometrical relations (4)

Uvedené vztahy obecně neplatí:

\[ \gamma_{yz} = \gamma_{zy}, \]
\[ \gamma_{zx} = \gamma_{xz}, \]
\[ \gamma_{xy} = \gamma_{yx}. \]

The **assumption of equality of shear stresses** is based on **approximative fulfillment** of moment equilibrium equations of differential element. Shear deformations are assumed to be equal in the same way.
Differentialequilibrium conditions

(1)

\[ \sigma_x' = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx, \quad \tau_{xy}' = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dy, ... \]
Diff. equilibrium conditions (2)

\[ \sigma_{x'} = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx, \quad \tau_{xy'} = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy, \ldots \]

\[
\sum F_{i,y} = (\sigma'_{x} - \sigma_x) \, dx \, dy + (\tau_{xy} - \tau'_{xy}) \, dx \, dz + (\tau_{xz} - \tau_{xz}') \, dx \, dy = 0
\]

\[
(\sigma_{x} - \sigma_x - \frac{\partial \sigma_x}{\partial x} \, dx) \, dy \, dz + (\tau_{xy} - \tau_{xy} - \frac{\partial \tau_{xy}}{\partial y} \, dy) \, dx \, dz + (\tau_{xz} - \tau_{xz} - \frac{\partial \tau_{xz}}{\partial z} \, dz) \, dx \, dy = 0
\]

After simplification:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0
\]
Diff. equilibrium conditions (3)

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z &= 0
\end{align*}
\]

where \( X, Y, Z \) are volume forces.
Material law (1)

Defines relations between stresses and strains.

Hooke law for 1D problem (simple tension/compression):

\[ \varepsilon_x = \frac{\sigma_x}{E} \]

\[ \varepsilon_x = \frac{\Delta L}{L} = \frac{\partial L}{\partial x} \]

\[ \sigma_x = \frac{F}{A} = E \varepsilon_x \]
Material law (2)

Hooke law for 3D problems:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right], \\
\varepsilon_y &= \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right], \\
\varepsilon_z &= \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right],
\end{align*}
\]

\[
\begin{align*}
\gamma_{yz} &= \frac{\tau_{yz}}{2G}, \\
\gamma_{xz} &= \frac{\tau_{xz}}{2G}, \\
\gamma_{xy} &= \frac{\tau_{xy}}{2G}
\end{align*}
\]
Conclusion

15 unknowns:
3 displacements \( u \)
6 stresses \( \varepsilon \)
6 strains \( \sigma \)

15 available equations:
6 geometrical relations
6 matrial law equations 3 equilibrium conditions
Compatibility conditions (1)

They define continuity of deformations – the continuous volume must remain to be continuous also after deformation.

They can be obtained from geometrical relations by elimination of displacements.

\[
\begin{align*}
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_z}{\partial z^2} &= \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \\
\end{align*}
\]
Compatibility conditions (2)

\[ \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{1}{2} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \]

\[ \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{1}{2} \left( -\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \]

\[ \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{1}{2} \left( +\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \]  (13)

For plane problems (2D, to be discussed later) it can be written:

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \]  (14)
Topic 2: Plane problems

2D plane structures supported and loaded in their middle plane will be discussed.

Two main variants of plane problem:

- plane stress
- plane strain
All **deformations** (strains) must lie in the $x - y$ plane:

\[ \varepsilon = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T \]  

(15)

The structure **can not** freely move in $z$ direction. As a result, there is $\sigma_z$ stress. $\sigma_z$:

\[ \sigma = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}\}^T \]  

(16)
Plane strain

All stresses lie in the $x - y$ plane:

$$\sigma = \{\sigma_x, \sigma_y, \tau_{xy}\}^T \quad (17)$$

The wall can freely move in $z$ direction, therefore there is on $\sigma_z$ stress but the $\varepsilon_z$ strain is nonzero:

$$\varepsilon = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}\}^T \quad (18)$$
Equilibrium equations

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\tau_{xy}}{\partial y} + X = 0 \]  
\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\sigma_y}{\partial y} + Y = 0 \]  

Geometrical relations

\[ \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \]
Plane problem – conclusions (2)

Material law: plane stress

\[
\begin{align*}
\sigma_x &= \frac{E}{1 - \mu^2} (\varepsilon_x + \mu \varepsilon_y) \\
\sigma_y &= \frac{E}{1 - \mu^2} (\varepsilon_y + \mu \varepsilon_x) \\
\tau_{xy} &= \frac{E}{2(1 - \mu)} \gamma_{xy}
\end{align*}
\]  

(21)  

(22)
Plane problem – conclusions (3)

Material law: plane strain

\[
\begin{align*}
\sigma_x &= \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-\mu) \varepsilon_x + \mu \varepsilon_y \right] \\
\sigma_y &= \frac{E}{(1+\mu)(1-2\mu)} \left[ \mu \varepsilon_x + (1-\mu) \varepsilon_y \right] \\
\tau_x &= \frac{E}{(1+\mu)(1-2\mu)} \gamma_{xy} \frac{1}{2} (1-\mu)
\end{align*}
\]  

(23)
Airy’s equation for plane problem

**Airy function** $F$ – describes stress state of plane problem if:

$$
\sigma_x = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xz} = -\frac{\partial^2 F}{\partial x \partial y}.
$$  \hspace{1cm} (24)

**Airy equation** – compatibility relation is (14) written with use of Airy function:

$$
\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0
$$  \hspace{1cm} (25)
Stress distribution on wall:

N \ldots beam theory,
S \ldots plane stress theory.
Forces in wall

\[ N_x = \sigma_x h \left[ \frac{N}{m} \right] \]
\[ N_y = \sigma_y h \left[ \frac{N}{m} \right] \]
\[ N_{xy} = \tau_{xy} h \left[ \frac{N}{m} \right] \]