Isoparametric finite elements

- Computation of stiffness matrix is done on element defined in natural coordinates \((\eta, \xi, \zeta = < -1, 1 >)\): line, rectangle, cube.

- Element in natural coordinates \((\eta, \xi, \zeta)\) is transformed to real finite in \((x, y, z)\) coordinates with use of shape functions \(N_i\).

- Shape functions are used also as approximation functions of unknown displacements.
Natural coordinates

- Lenght of element side is $1 + 1 = 2$

- Here we will use natural coordinates $s, t$ (in textbooks are often called $\xi, \eta$)
Relation between $x$ and $s$:

$$x = \sum_i N_i(s) x_i = N_1 x_1 + N_2 x_2 + \ldots,$$

(1)

where $N_i$ are shape functions.
Shape functions

In order to use equation (2), shape functions $N_i$ have to have these values:

- in the $i$ point .... 1,

- in any other point ... 0.
Deformations by shape functions

\[ u = \sum_i N_i(s) \, u_i, \quad (2) \]

For this particular finite element:

\[ u = N_1(s) \, u_1 + N_2(s) \, u_2 + N_3(s) \, u_3 \quad (3) \]
Truss element (1)

Natural and real coordinates:

\[ \begin{align*}
-1 & \quad 0 & \quad +1 \\
\hline
1 & \quad 1
\end{align*} \]

Unknown displacements:

\[ \begin{align*}
x_1 & \quad u_1 & \quad x_2 & \quad u_2 & \quad x_3 & \quad x \\
\hline
\end{align*} \]

There are unknown displacement on element: \( u = u_1, u_2, u_3^T \)
Truss element (2)

Shape functions:

\[ N_1 = -\frac{s(1-s)}{2} \]
\[ N_2 = \frac{(1-s^2)}{s(1+s)} \]
\[ N_3 = \frac{s(1+s)}{2} \]

Thus:

\[ x = N_1 x_1 + N_2 x_2 + N_3 x_3 = -\frac{s(1-s)}{2} x_1 + (1-s^2) x_2 + \frac{s(1+s)}{2} x_3 \]
Truss element (3)

Shape functions:

\[ x = N_1 x_1 + N_2 x_2 + N_3 x_3 = -\frac{s(1-s)}{2} x_1 + (1-s^2) x_2 + \frac{s(1+s)}{2} x_3 \]  

(6)

In matrix form:

\[ \{x\} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]  

(7)
Truss element (4)

Approximation of displacements $u$ (the same shape functions like for $x$):

$$u = \sum_i N_i(s) u_i,$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 = -\frac{s(1 - s)}{2} u_1 + (1 - s^2) u_2 + \frac{s(1 + s)}{2} u_3$$

In matrix form:

$$\{u\} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (9)$$

$$\{u\} = \begin{bmatrix} -\frac{s(1-s)}{2} & (1 - s^2) & \frac{s(1+s)}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
Truss element (5)

Relative deformations:

\[ \varepsilon = \partial u \] \hspace{1cm} (10)

thus:

\[ \varepsilon = \frac{\partial u}{\partial x} = \left\{ \frac{\partial}{\partial x} \right\} \left[ -\frac{s(1-s)}{2} (1 - s^2) \frac{s(1+s)}{2} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \] \hspace{1cm} (11)

after derivation:

\[ \varepsilon = \left[ \frac{\partial (-s(1-s)/2)}{\partial x} \frac{\partial (1-s^2)}{\partial x} \frac{\partial s(1+s)/2}{\partial x} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \] \hspace{1cm} (12)
Truss element (6)

Relative deformations:

\[
\varepsilon = \begin{bmatrix}
\frac{\partial (-s(1-s))}{\partial x} & \frac{\partial (1-s^2)}{\partial x} & \frac{\partial s(1+s)}{2} \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

in short form \( \varepsilon = B u \):

\[
\varepsilon = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

(13)

**Problem:** Equations for \( \varepsilon \) includes \( \frac{\partial N_i}{\partial x} \), but \( N_i \) is function of \( s \).
Truss element (7)

From mathematics:

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial s} \frac{\partial s}{\partial x} \quad (14)$$

Equation (2):

$$x = \sum_i N_i(s) \ x_i$$

After derivation of (2):

$$\frac{dx}{ds} = \sum_i \frac{\partial N_i}{\partial s} \ x_i = J \quad (15)$$

The member (usually matrix) $J$ is Jakobian.
Truss element (8)

From equation

\[ \frac{dx}{ds} = J \]

we can get \( ds \):

\[ ds = \frac{1}{J} \, dx \]  \hspace{1cm} (16)

**Note:** In general problems the member \( \frac{1}{J} \) is an inverse of Jacobian matrix: \( J^{-1} \).
Truss element (9)

From equations

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial s} \frac{\partial s}{\partial x}$$

and

$$ds = \frac{1}{J} \, dx$$

one can get:

$$\frac{\partial N_i}{\partial x} = J \frac{\partial N_i}{\partial s} \Rightarrow \frac{\partial N_i}{\partial s} = \frac{\partial N_i}{\partial x} \frac{1}{J}$$  \hspace{1cm} (17)
Truss element (10)

From equation (17) it is possible to get $B$:

$$B = \left[ \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \right] = \frac{1}{J} \left[ \frac{\partial N_1}{\partial s} \quad \frac{\partial N_2}{\partial s} \quad \frac{\partial N_3}{\partial s} \right]$$  \hfill (18)

Thus:

$$\{\varepsilon\} = \frac{1}{J} \left[ \frac{\partial N_1}{\partial s} \quad \frac{\partial N_2}{\partial s} \quad \frac{\partial N_3}{\partial s} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\}$$  \hfill (19)
Truss element (11)

If we know $\varepsilon$:

$$\{\varepsilon\} = \frac{1}{J} \left[ \frac{\partial N_1}{\partial s} \quad \frac{\partial N_2}{\partial s} \quad \frac{\partial N_3}{\partial s} \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

it is possible to write equation for potential energy of internal forces:

$$\Pi_i = \frac{1}{2} \int_V \varepsilon^t D \varepsilon dV, \quad (20)$$

and equaito for stiffness matrix of finite element:

$$K = A \int_{x_1}^{x_3} \varepsilon^t D \varepsilon dx. \quad (21)$$
Truss element (12)

Equation (22) can be integrated analytically here. It is not possible for more complicated problems (2D, 3D) thus it is necessary to utilize numerical integration. In many cases Gauss formula is used:

\[
K = A \sum_{i=1}^{m} \varepsilon^t D \varepsilon w_i, \tag{22}
\]

where \(w_i\) is weight of integrational point, \(m\) is number of integrational points.
Isoparametric elements for plane problems (1)

Shape functions:

\[ N_i(\xi, \eta) = \frac{1}{4}(1 + \xi \xi_i)(1 + \eta \eta_i), \quad (23) \]
Izoparametric element for plane problems (2)

Shape functions for corners:

\[ N_i(\xi, \eta) = \frac{1}{4} (1 + \xi \xi_i) (1 + \eta \eta_i) (\xi \xi_i + \eta \eta_i - 1), \]  

and for mid-sides:

\[ N_i(\xi, \eta) = \frac{\xi_i^2}{2} (1 + \xi \xi_i) (1 - \eta^2) + \frac{\eta^2}{2} (1 - \xi \xi_i)(1 - \eta^2). \]